Mathematical Undecidability and Quantum Randomness

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COLLABORATORS

Theory

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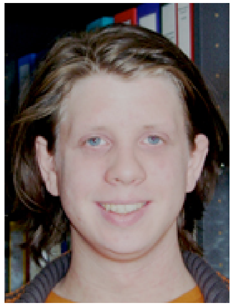

Peter
Klimek

Caslav
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Experiment

Robert
Prevedel

Markus
Aspelmeyer

Anton
Zeilinger

What is the origin of quantum randomness?

Quantum randomness is a manifestation of mathematical undecidability.

A proposition is undecidable within a set of axioms,
if it can neither be proved nor disproved within the set.

Undecidability arises if a proposition, together with the axioms, contains more information than the set of axioms itself.

Boolean functions of a binary argument

$$x \in \{0,1\} \rightarrow y = f(x) \in \{0,1\}$$

Single bit axiom: $f(0) = 0$

Proposition to be proved: $f(0) = f(1)?$

Undecidable!
Requires two bits, but the axiom contains only one.

Similarly, $f(1) = ?$ is undecidable within the axiom.

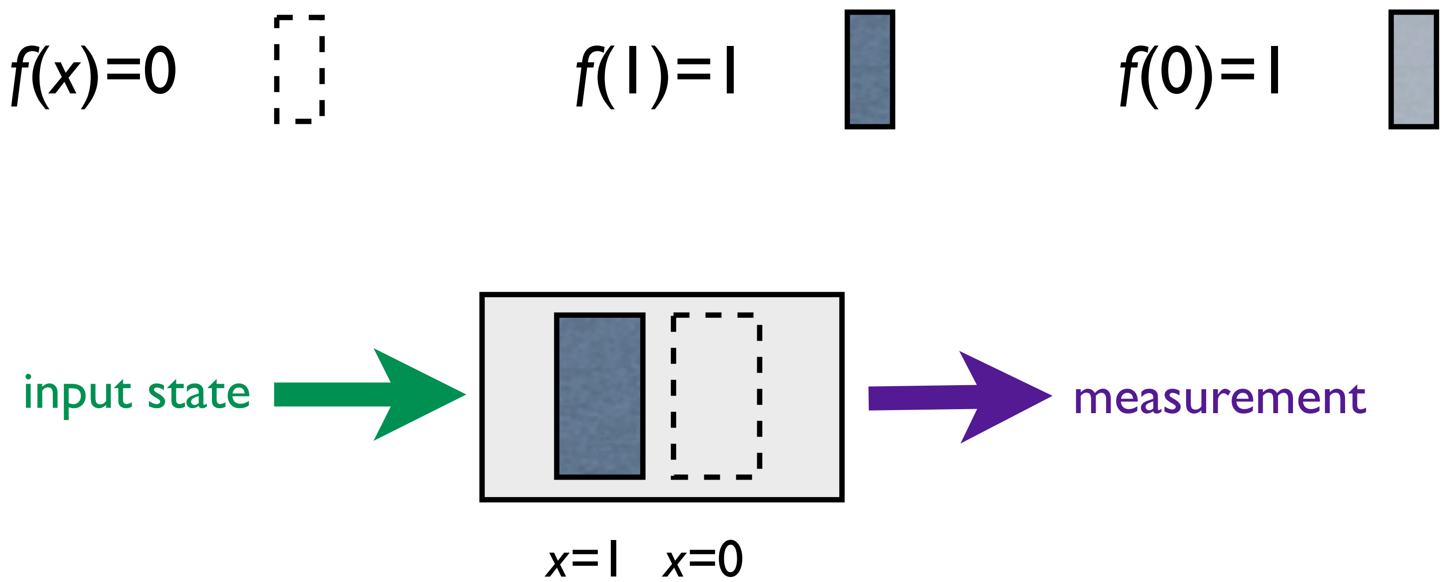
Given limited information resources, propositions which cannot be simultaneously ascribed definite truth values are logically complementary.

(A) $f(0) = 0$

(B) $f(1) = 0$

(C) $f(0) = f(1)$

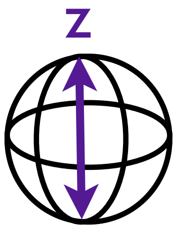
A black box encodes the Boolean functions



$$U = \sigma_x^{f(0)} \sigma_z^{f(1)}$$

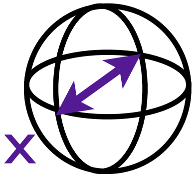
QUANTUM AND LOGICAL COMPLEMENTARITY

$|z\pm\rangle \rightarrow \boxed{U = \sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$



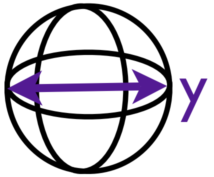
$f(0) = ?$

$|x\pm\rangle \rightarrow \boxed{U = \sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$



$f(1) = ?$

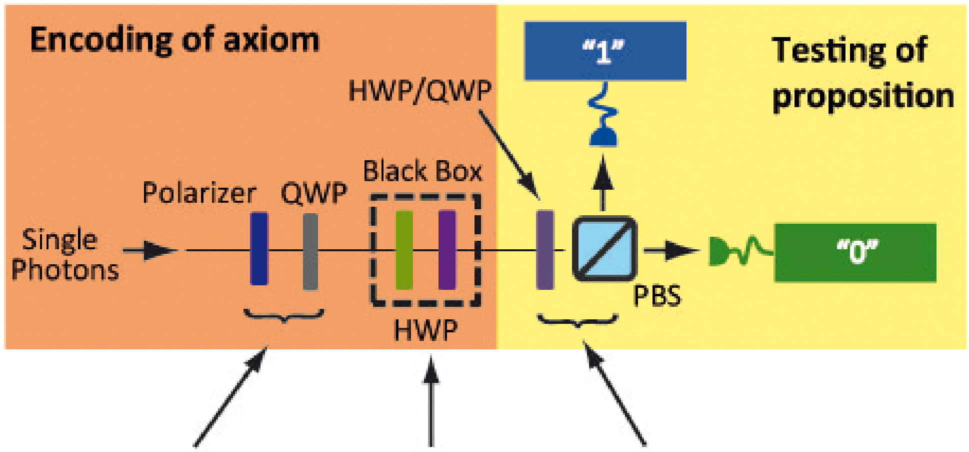
$|y\pm\rangle \rightarrow \boxed{U = \sigma_x^{f(0)}\sigma_z^{f(1)}} \rightarrow$



$f(0) = f(1)?$

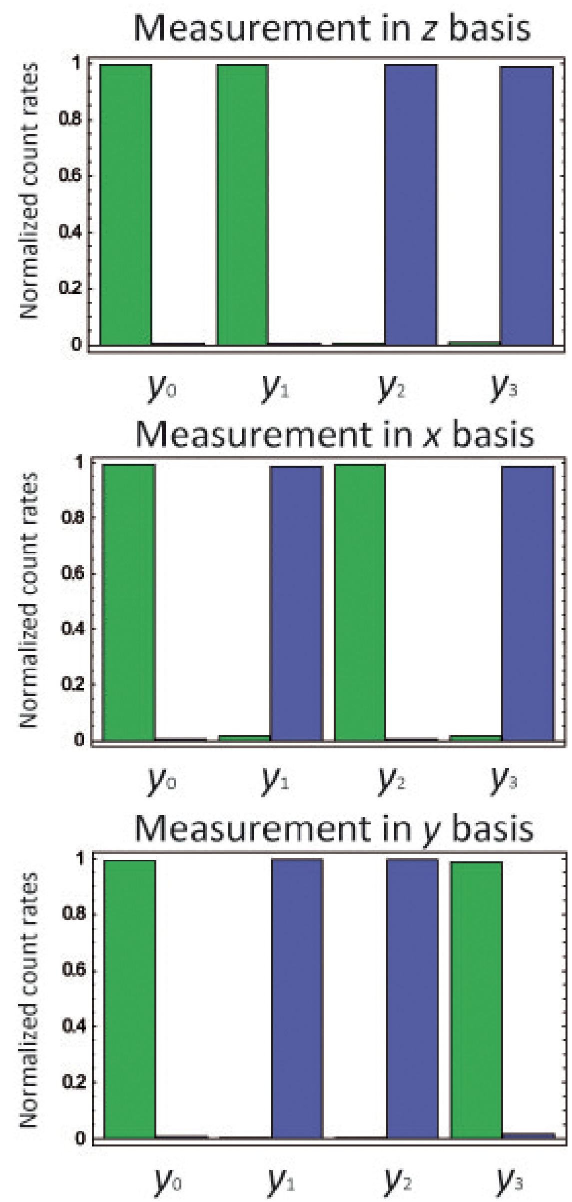
Quantum complementary states answer logically complementary questions.

EXPERIMENTAL ILLUSTRATION

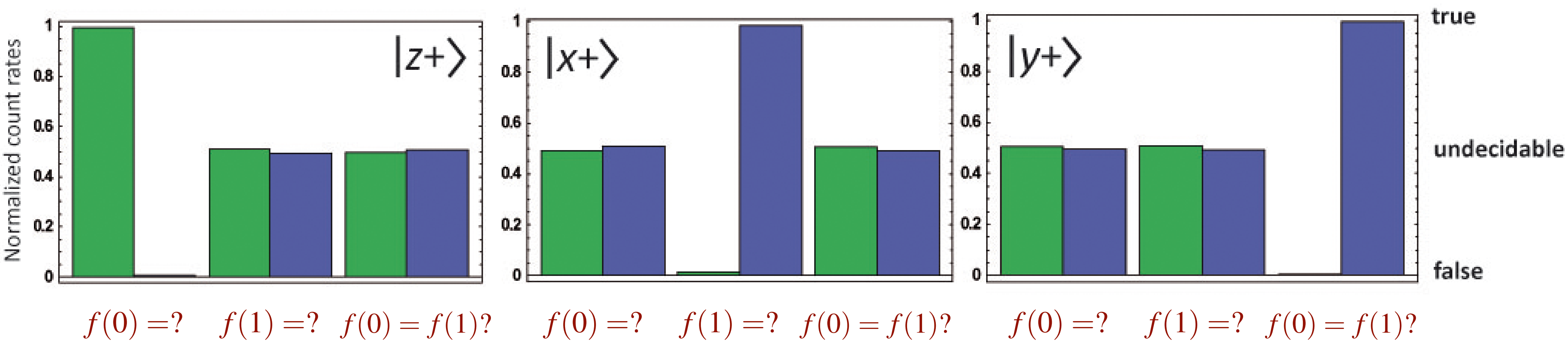


Preparation	Black Box	Measurement
$\lvert z+\rangle$ = Polarizer 0°	Id → no waveplate	$\langle z\rvert$: no waveplate
$\lvert x+\rangle$ = Polarizer +45°	σ_x → HWP@45°	$\langle x\rvert$: HWP@22,5°
$\lvert y+\rangle$ = Polarizer 0° + QWP@45°	σ_z → HWP@0°	$\langle y\rvert$: QWP@45°
	$\sigma_x\sigma_z$ → HWP@45° + HWP@0°	

x	y_0	y_1	y_2	y_3
0	0	0	1	1
1	0	1	0	1



Mathematics/Logic	Quantum Physics
Proposition	Measurement
Axioms	State
Decidability/Undecidability	Definiteness/Randomness



Mathematical reasoning for irreducible quantum randomness

Quantum systems have limited information content

Holevo, Zeilinger

Measurements are identified with propositions

When a quantum state is measured in a complementary basis the results must contain no information about the truth of the undecidable proposition. They must be random.

Encoded in stabilizer states

Partial undecidability

What is the truth value?

$$|GHZ\rangle = (|z+\rangle_1|z+\rangle_2|z+\rangle_3 + |z-\rangle_1|z-\rangle_2|z-\rangle_3)/\sqrt{2}$$

$$f_1(0) + f_1(1) + f_2(0) + f_2(1) + f_3(1) = 1 \qquad \sigma_y \otimes \sigma_y \otimes \sigma_x$$

$$f_1(0) + f_1(1) + f_2(1) + f_3(0) + f_3(1) = 1 \qquad \sigma_y \otimes \sigma_x \otimes \sigma_y$$

$$f_1(1) + f_2(0) + f_2(1) + f_3(0) + f_3(1) = 1 \qquad \sigma_x \otimes \sigma_y \otimes \sigma_y$$

Logic: $f_1(1) + f_2(1) + f_3(1) = 1$

Quantum: $f_1(1) + f_2(1) + f_3(1) = 0 \qquad \sigma_x \otimes \sigma_x \otimes \sigma_x \quad !$

This new viewpoint suggests that the incompleteness phenomenon discovered by Gödel is natural and widespread rather than pathological and unusual.

Chaitin, *Gödel's Theorem and Information*.

Physical systems have limited information content.

Measurements can be identified with propositions.

Quantum randomness is a manifestation of mathematical undecidability.

THEORY SETUP